LONGITUDINAL FLOW OF A SECOND GRADE FLUID IN A CYLINDER

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ABSTRACT: *This paper is considered to analyze the governing equations for velocity and shear stress in the longitudinal flow of a second grade fluid in an infinite cylinder. Hankel transform have been used to obtain the solution of governing equations of shear stress and velocity. The results have been computed for several values of the parameters involved. The graphs of the comparison between the shear stress of the second grade fluid and the Newtonian fluids are drawn for different values of parameters in this study.*

Key Words: Cylinder, Secod Grade Fluid, Analytic solution, Netwonian fluid.

1. INTRODUCTION

Considerable attention has been paid by the researchers to solve problems of non-Newtonian fluids due to their industrial and technological applications. Several models have been developed to describe the behavior of non-Newtonian fluids. Second grade fluids (a subclass of differential type fluids) have been studied successfully in different kind of motion by many scientists and researchers. Ting [1] had studied the unsteady flows of a second grade fluid in a bounded region. He was the first one who discussed the unsteady flows of second grade fluids. Rajagopal [2] analyzed the equations of motion of second grade fluid. Rajagopal and Gupta [3] had engaged with the equations of motion of second grade fluid. Hayat *et al.* [4] conducted a study on the unidirectional flow of a fluid and unstable non - Newtonian second grade fluid. Xu and Tan [5] investigated a theoretical analysis of the stress field, velocity field and vortex sheet of fractional second grade fluid with fractionated anomalous diffusion. Tan *et al*. [6] obtained an exact solution of the Couette flow of the fraction second grade fluid.

Hankel transforms have been repeatedly used by reseachers to obtain analytical results for such problems. Among many others, Siddique *et al.* [7] solved the problems concerning the longitudinal flow as well as rotational flow of a Newtonian fluid. Mehmood *et al.* [8] found the velocity field as well as the associated shear stress related to the problems of the longitudinal oscillation of second grade fluid between two infinite coaxial circular cylinders. Jamil [9] established exact analytic solutions for helical flows of a second grade fluid between two infinite coaxial cylinders. Rakhi *et al*. [10] had studied the rotational oscillation of generalized second grade fluid between two infinite coaxial circular cylinders. Kamran *et al*. [11] obtained the exact analytic solutions for the flow of a generalized second grade fluid in an annular region between two infinite coaxial cylinders.

Erdogan and Imrak [12] conducted a study about some unsteady flows of second grade fluid where the unsteady unidirectional flows are considered. Bandelli [13] had carried out a research on the heated boundaries of the unsteady unidirectional flows of second grade fluids. Ali et al. [14] found the unsteady oscillatory flow of an incompressible second grade fluid in a cylindrical tube with large wall suction is studied analytically. Kamran *et al.* [15] obtained the velocity field and the adequate shear stress corresponding to the unsteady flow of second grade fluid with fractional calculus due to a linearly accelerating circular cylinder. Mehmood *et al.* [16] calculated the velocity as well as the shear stress relating to the problems of unsteady flow of generalized second grade fluids through a constantly accelerating circular cylinder.

This work is an attempt to find the solutions of unsteady longitudinal flow of second grade fluid in cylindrical domain. The results for velocity profile and shear stresses have been obtained by using computer algebraic system (CAS) MATHCADE. The influence of material parameter and time for different values has been observed. Comparison of the results of shear stress of second grade fluid is made with those of Newtonian fluid.

2. MATHEMATICAL ANALYSIS:

Consider longitudinal flow of a second grade fluid in an infinite cylinder moving with velocity V, with no body force. The initial condition is at t = 0, V(r, t) = 0 and the boundary condition is V(r, t) = v t.

The velocity vector is V = (v_r , v_θ , v_z), as there is no flow along v_r and v_θ, so $v_r = 0$, $v_\theta = 0$ and flow is along v_z only.

$\Rightarrow v_z \neq 0$ and $V = v_z(r, t)$

The constitutive equation for the second grade fluid is

$$T = -pI + \mu\mathbf{A}_1 + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A}_1^2 \tag{1}$$

and p = 0 (Hydrostatic pressure is being neglected).
The equation (1) becomes:

$$T = \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2} \tag{2}$$

L=grad V , $\mathbf{A_1} = L + L^T$

A_2 is the kinematic tensor defined by

$$\mathbf{A_2 = \frac{dA_1}{dt} + A_1L + L^TA_1} \tag{3}$$

where d/dt denotes the material time derivative.

Putting the values of $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_1^2}$ in equation (2)

$$\boldsymbol{T} = \begin{pmatrix} (\alpha_1+\alpha_2)\left(\frac{\partial v_z}{\partial r}\right)^2 & 0 & \mu\frac{\partial v_z}{\partial r}+\alpha_1\frac{\partial^2 v_z}{\partial t\partial r} \\ 0 & 0 & 0 \\ \mu\frac{\partial v_z}{\partial r}+\alpha_1\frac{\partial^2 v_z}{\partial t\partial r} & 0 & (\alpha_1+\alpha_2)\left(\frac{\partial v_z}{\partial r}\right)^2 \end{pmatrix} \quad (4)$$

Also $\boldsymbol{T} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$ (5)

Comparing equation (4) and equation (5), we get

$$\tau_{rz} = \mu\frac{\partial v_z}{\partial r}+\alpha_1\frac{\partial^2 v_z}{\partial t\partial r}$$

Taking $v_z = v$,

$$\tau_{rz} = \left(\mu+\alpha_1\frac{\partial}{\partial t}\right).\frac{\partial v}{\partial r}$$

$$\tau(r,t) = \left(\mu+\alpha_1\frac{\partial}{\partial t}\right).\frac{\partial}{\partial r}v(r,t) \quad (6)$$

which gives governing equation for shear stress.

The equations of motion

Div $\bar{T}$ + ρb = ρa (7)

Since the body force b=0 and $v_z = v$,

The equation (7) yields:

$$\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\tau_{rz} = \rho\frac{\partial v}{\partial t}$$

substituting the equation for shear stress,

$\frac{\mu}{\nu}=\rho$ and $\alpha_1 = \alpha\rho$

$$\Rightarrow \frac{\partial v_{(r,t)}}{\partial t} = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v_{(r,t)}\left(\nu+\alpha\frac{\partial}{\partial t}\right) \quad (8)$$

which is the governing equation for velocity.

3. Analytical Solution of the Governing Equations

The equation (6) and equation (8) are solve analytically by using Hankel Transform.

Let $f(r) = \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}$

Apply Hankel transform

$$\int_0^R r f(r) J_0(r,r_n)dr = \int_0^R r\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)J_0(r,r_n)dr$$

$$= \int_0^R r\frac{\partial^2 v}{\partial r^2}J_0(r,r_n)dr + \int_0^R \frac{\partial v}{\partial r}J_0(r,r_n)dr$$

$$= \left[\frac{\partial v}{\partial r}.r\,J_0(r,r_n)\right]_0^R - \int_0^R \frac{\partial v}{\partial r}.\frac{d}{dr}[(r.J_0(r,r_n)]dr$$

$$+\int_0^R \frac{\partial v}{\partial r}J_0(r,r_n)dr$$

$\therefore$ $J_0(R, r_n) = 0$

$$\int_0^R r\,f(r)J_0(r,r_n)dr$$

$= r_n v$

R=

$$J_1(R,r_n) - r_n\int_0^R v.J_1(r,r_n)dr - r_n\int_0^R v.r\frac{d}{dr}[J_1(r,r_n)]dr \quad (10)$$

$$\therefore \frac{d}{dr}J_1(r,r_n) = J_0(r,r_n)r_n - \frac{1}{r}J_1(r,r_n)$$

Substituting the value in equation (10), we get

$$\int_0^R r\,f(r)J_0(r,r_n)dr$$

$$= r_n vRJ_1(R,r_n) - r_n\int_0^R v.r\left[J_0(r,r_n).r_n - \frac{1}{r}J_1(r,r_n)\right]dr$$

$$\int_0^R rf(r)J_0(r,r_n)dr = r_n vRJ_1(r,r_n) - r_n^2\int_0^R v.r\,J_0(r,r_n)dr$$

$$\int_0^R rf(r)J_0(r,r_n)dr = r_n vRJ_1(r,r_n) - r_n^2 v_H(r_n,t)$$

Apply the boundary condition $v(r,t) = vt$, we get

$$\int_0^R rf(r)J_0(r,r_n)dr = r_n vtRJ_1(r,r_n) - r_n^2 v_H(r_n,t)$$

Putting values in equation (8), we get

$$\frac{\partial v_H(r_n,t)}{\partial t} = \left(\nu+\alpha\frac{\partial}{\partial t}\right)(r_n vtR\,J_1(R,r_n) - r_n^2 v_H(r_n,t)).$$

$$\frac{\partial v_H(r_n,t)}{\partial t} = \nu\, r_n\, vtR\,J_1(R,r_n) - \nu\, r_n^2 v_H(r_n,t) + \alpha\, r_n vR\,J_1(R,r_n) - \alpha\, r_n^2\frac{\partial}{\partial t}v_H(r_n,t)).$$

$$\frac{\partial v_H(r_n,t)}{\partial t} + \frac{\nu r_n^2}{1+\alpha r_n^2}v_H(r_n,t)$$

$$= \frac{1}{1+\alpha r_n^2}\left[\nu\, r_n v\, tR\,J_1(R,r_n) + \alpha\, r_n\, vR\,J_1(R,r_n)\right] \quad (11)$$

which is linear differential equation.

I.F. $= e^{\int\frac{\nu r_n^2}{1+\alpha r_n^2}dt} = e^{\frac{\nu r_n^2 t}{1+\alpha r_n^2}}$

$$v_H(r_n,t) = vR\,J_1(R,r_n)\left[\frac{t}{r_n} - \frac{1}{\nu r_n^3}\right] + c.e^{\frac{-\nu r_n^2 t}{1+\alpha r_n^2}} \quad (12)$$

Applying initial condition in equation (4.3.4) (9)

t = 0, $v_H(r_n,t) = 0$

$$\boxed{c = vRJ_1(R,r_n)\frac{1}{\nu r_n^3}}$$

Substituting the value of c in equation (12), we get

$$\Rightarrow v_H(r_n,t) = J_1(R,r_n)\, vR\left[\frac{t}{r_n} - \frac{1}{\nu\, r_n^3} + \frac{1}{\nu\, r_n^3}\, e^{\frac{-\nu\, r_n^2 t}{1+\alpha r_n^2}}\right] \tag{13}$$

Apply inverse Hankal transform in equation (13)

$$v(r,t) = vt - \frac{2v}{\nu\ R}\sum_{n=1}^{\infty}\frac{J_0(r,r_n)}{J_1(R,r_n)}\cdot\frac{1}{r_n^3}\left[1 - e^{\frac{-\nu\, r_n^2 t}{1+\alpha r_n^2}}\right] \tag{14}$$

which gives velocity of second grade fluid.

The governing equation for shear stress is

$$\tau(r,t) = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\cdot\frac{\partial}{\partial r}v(r,t)$$

substituting the value of $v(r,t)$ from equation (14) and $\frac{d}{dr}(J_0(r,r_n)) = -J_1(r,r_n)\cdot r_n$, we get

$$\tau(r,t) = \frac{2\rho v}{R}\sum_{n=1}^{\infty}\frac{J_1(r,r_n)}{r_n^2 J_1(R,r_n)}\left[1 - e^{\frac{-\nu\, r_n^2 t}{1+\alpha r_n^2}}\right] + \frac{2\propto\rho v}{\nu\ R}\sum_{n=1}^{\infty}\frac{J_1(r,r_n)}{r_n^2 J_1(R,r_n)}\, e^{\frac{-\nu\, r_n^2 t}{1+\alpha r_n^2}}\left(\frac{\nu\, r_n^2}{1+\alpha r_n^2}\right)$$

For α=1, we get the result

$$\tau(r,t) = \frac{2\rho v}{R}\sum_{n=1}^{\infty}\frac{J_1(r,r_n)}{r_n^2 J_1(R,r_n)}\left[1 - \left(\frac{1}{1+\alpha r_n^2}\right)e^{\frac{-\nu\, r_n^2 t}{1+\alpha r_n^2}}\right] \tag{15}$$

which gives shear stress f or second grade fluid.

4. NUMERICAL RESULTS

The results of solutions for the equations (13) and (15) are obtained to elaborate the effects of physical parameters involved in these equations, on velocity profile and shear stress. Then some properties of the fluid motion as well as the influences of parameters on the behavior of the fluid motion are shown by graphs. Then graphs of ν (r, t) for different values of parameters are drawn. By changing, the values of parameters, we get different graphs.

Finally, the graphs of the comparison between the shear stress of the second grade fluid and the Newtonian fluids are drawn and graphs for different values of parameters are shown in figures. By changes the values of parameters, we get different graphs.

In the next pages we draw graphs of ν (r, t) for different values of parameters. We get different types of graphs as shown in next pages. Then we draw graphs of the comparison between the shear stress of the second grade fluid and the Non- Newtonian fluids. The units of parameters into figures are SI units.

Velocity and Shear Stress Graphs of Second Grade Fluid

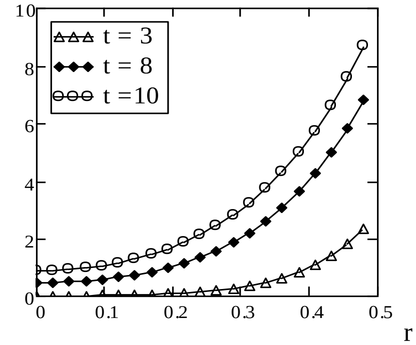

Figure-1: Profile of velocity V(r, t) graph for R= 0.5, ν = 1, μ=2.916, α = 0.003, ν =0.003 and for different values of t.

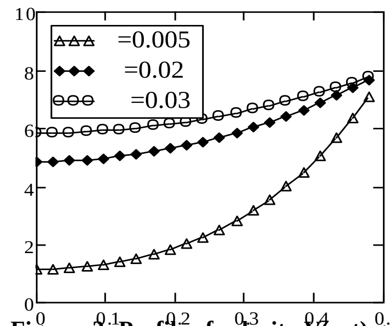

Figure- 2: Profile of velocity V(r, t) graph for R= 0.5, ν = 1,μ=2.916, α = 0.003,, t = 8 and for different values of ν .

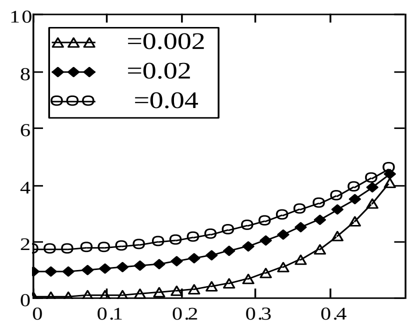

Figure-3. Profile of velocity V(r, t) graph for R= 0.5, ν = 1, μ=2.916, ν =0.003, t = 5 and for different values of α.

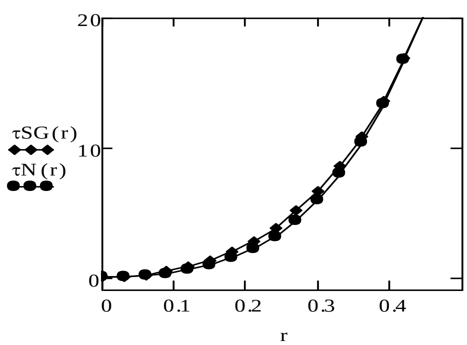

Figure- 4: Comparison of shear stress τ(r, t) graph R= 0.5, ν = 1, μ=2.916, α = 0.004, ν =0.005, ρ=600, t = 3

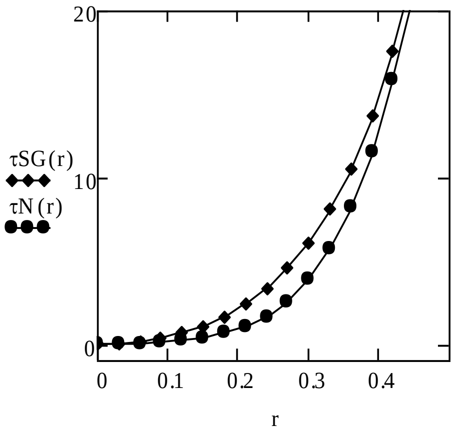

Figure-5: Comparison of shear stress τ(r, t) graph R= 0.5, ν = 1, μ=2.916, α = 0.004, ν =0.003, ρ=800, $t = 3$

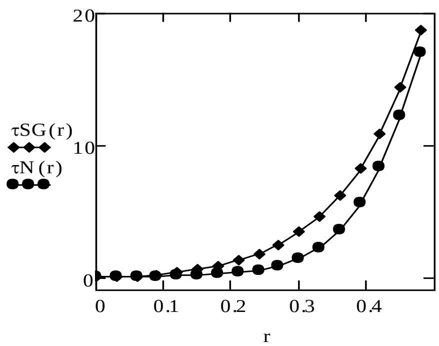

Figure-6: Comparison of shear stress τ(r, t) graph R= 0.5, ν = 1, μ=2.916, α = 0.004, ν =0.0006, ρ=600, $t = 10$

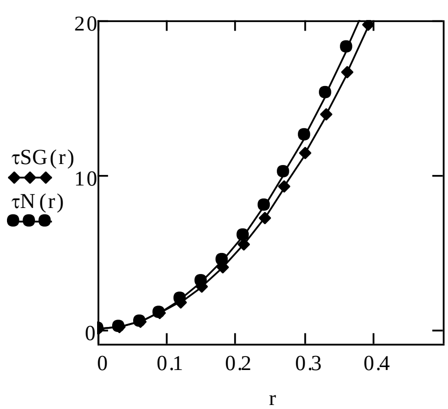

Figure-7: Comparison of shear stress τ(r, t) graph R= 0.5, ν = 1, μ=2.916, α = 0.003, ν =0.003, ρ=600, $t = 3$

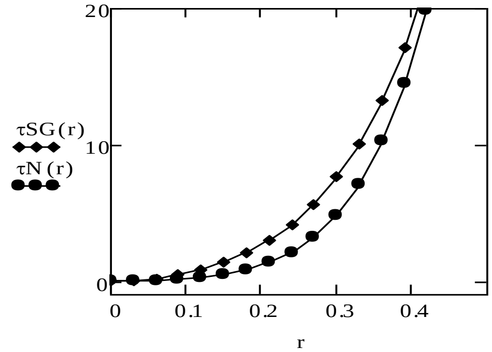

Figure-8: Comparison of shear stress τ(r, t) graph for R= 0.5, ν = 1, μ=2.916, α = 0.003, ν =0.003, ρ=600, $t = 3$

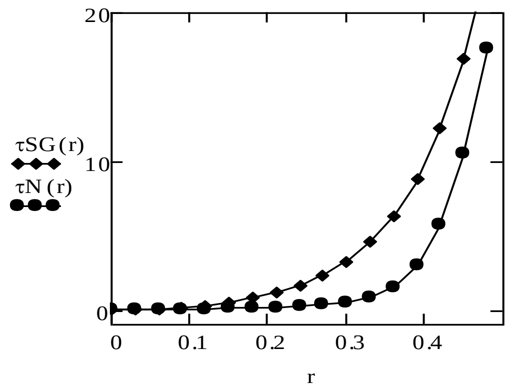

Figure-9: Comparison of shear stress τ(r, t) graph for R= 0.5, ν = 1, μ=2.96, α = 0.004, ν =0.003, ρ=900, $t = 1$

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